stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 110/78 APRIL
(DEPARTMENT OF PURE MATHEMATICS)

A. SCHRIJVER & P.D. SEYMOUR

SOLUTION OF TWO FRACTIONAL PACKING PROBLEMS OF LOVÁSZ

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme: 05A05, 05B40, 90C05, 90C10

Solution of two fractional packing problems of Lovász *)

by

A. Schrijver & P.D. Seymour **)

ABSTRACT

Lovász asked whether the following is true for each hypergraph H and natural number k:

(*) if $\nu_k(H') = k.\nu^*(H')$ holds for each hypergraph H' arising from H by multiplication of points, then $\nu_k(H) = \tau_k(H)$;

(**) if $\tau_k(H') = k.\tau^*(H')$ holds for each hypergraph H' arising from H by removing edges, then $\tau_k(H) = \nu_k(H)$.

We prove and generalize assertion (*) and give a counterexample to (**).

KEYWORDS & PHRASES: *fractional packing, duality, linear programming*

---

*) This report will be submitted for publication elsewhere

**) P.D. Seymour, Merton College, Oxford, England

1. INTRODUCTION

Let $H = (X,\mathcal{E})$ be a hypergraph (i.e. X is a finite set and $\mathcal{E}$ is a family of subsets of X; the elements of X and the sets in $\mathcal{E}$ are called the *points* and *edges* of H, respectively).

Let $\nu_k(H)$ be the maximum number of edges (possibly taking edges repeated) such that no point is contained in more than k of the chosen edges; that is

$$\nu_k(H) = \max\{\sum_{E\in\mathcal{E}} m(E) \mid m:\mathcal{E} \to \mathbb{Z}_+;\ \sum_{E\ni x} m(E) \le k \text{ for each } x \in X\}. \tag{1}$$

[$\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the sets of nonnegative integers and real numbers, respectively.] Let $\tau_k(H)$ be the minimum number of points (again, possibly with points repeated) such that no edge contains fewer than k of the chosen points; in formula

$$\tau_k(H) = \min\{\sum_{x\in X} t(x) \mid t:X \to \mathbb{Z}_+;\ \sum_{x\in E} t(x) \ge k \text{ for each } E \in \mathcal{E}\}. \tag{2}$$

(We allow H to have empty edges, so these numbers may be infinite.) $\nu_1(H)$ and $\tau_1(H)$ are usually abbreviated to $\nu(H)$ and $\tau(H)$, respectively. The duality theorem of linear programming implies that the numbers

$$\nu^*(H) = \max\{\sum_{E\in\mathcal{E}} m(E) \mid m:\mathcal{E} \to \mathbb{R}_+;\ \sum_{E\ni x} m(E) \le 1 \text{ for each } x \in X\} \tag{3}$$

and

$$\tau^*(H) = \min\{\sum_{x\in X} t(x) \mid t:x \to \mathbb{R}_+;\ \sum_{x\in E} t(x) \ge 1 \text{ for each } E \in \mathcal{E}\} \tag{4}$$

are equal. Since the linear programs defining ν^* and τ^* have rational optimal solutions it follows that

$$\max_k \frac{\nu_k(H)}{k} = \nu^*(H) = \tau^*(H) = \min_k \frac{\tau_k(H)}{k}. \tag{5}$$

Note that for all k and ℓ:

$$\nu(H) \le \frac{\nu_k(H)}{k} \le \frac{\nu_{k\ell}(H)}{k\ell} \le \nu^*(H) = \tau^*(H) \le \frac{\tau_{k\ell}(H)}{k\ell} \le \frac{\tau_k(H)}{k} \le \tau(H). \tag{6}$$

A large part of the previous and present work on this examines to what extent the equality of certain terms in this series of inequalities implies the equality of other terms.

First recall the following definitions. *Removing a point* x means that we replace X by $X\setminus\{x\}$ and remove all edges from $\mathcal{E}$ containing x; the term *removing an edge* speaks for itself. *Multiplying a point* x by $k \geq 0$ means that we replace x by k new points $x_1,\ldots,x_k$, at the same time replacing each edge E containing x by the new edges $(E\setminus\{x\})\cup\{x_1\},\ldots,(E\setminus\{x\})\cup\{x_k\}$. So multiplying x by 0 agrees with removing x.

LOVÁSZ [4] proved:

(7) if $\nu(H') = \nu^*(H')$ holds for each hypergraph H' obtained from H by removing points then $\nu(H) = \tau(H)$;

and

(8) if $\tau(H') = \tau^*(H')$ holds for each hypergraph H' obtained from H by removing edges then $\nu(H) = \tau(H)$.

The following result of BERGE [1] is a sharpening of (8):

(9) if $\tau_2(H') = 2\tau(H')$ holds for each hypergraph H' obtained from H by removing edges then $\nu(H) = \tau(H)$.

LOVÁSZ [6] showed that under a stronger inheritance a weaker assumption in (7) is possible:

(10) if $\nu_2(H') = 2\nu(H')$ holds for each hypergraph H' obtained from H by multiplication of points then $\nu(H) = \tau(H)$.

We may replace in (9) and (10) the indices 2 by any $\ell \geq 2$. LOVÁSZ [7] wondered whether the following assertions, generalizing (7) and (8) respectively, would be true for each natural number k:

(11) if $\nu_k(H') = k\nu^*(H')$ holds for each hypergraph H' arising from H by multiplication of points then $\nu_k(H) = \tau_k(H)$,

and

(12) if $\tau_k(H') = k\tau^*(H')$ holds for each hypergraph H' arising from H by removing edges then $\nu_k(H) = \tau_k(H)$.

For $k = 1$ they follow from (10) and (8), respectively, and LOVÁSZ [5] proved them for $k = 2$. In [7] LOVÁSZ proved (12) for the case $k = 3$. Here we shall prove (11) for each integer k, and disprove (12) for $k = 60$. More generally, we shall prove:

(13) if $k\nu^*(H')$ is an integer for each hypergraph H' arising from H by multiplication of points then $k\nu^*(H) = \tau_k(H)$.

This was proved for $k = 1$ and $k = 2$ by LOVÁSZ (cf. [7]). By straightforwardly adapting the method of proof used by LOVÁSZ [6] to prove (10) the following generalization of both (10) and (11) can be proved.

(14) If $\nu_{2k}(H') = 2\nu_k(H')$ for each hypergraph H' arising from H by multiplication of points then $\nu_k(H) = \tau_k(H)$.

Again, we may replace in (14) the index 2 by an arbitrary $\ell \geq 2$.

We first give, in section 2, a counterexample to (12). Section 3 contains the proofs and section 4 some final remarks. For a survey of examples and applications of these results we refer to LOVÁSZ [7].

2. A COUNTEREXAMPLE

The following hypergraph $H = (X,E)$ is a counterexample to (12) in the case $k = 60$. Let

$$X = \{1,2,3,4,5,6,7,8,9\}$$

and

$$E = \{E_1,E_2,E_3,E_4,E_5,E_6,E_7\}, \text{ where}$$
$$E_1 = X\backslash\{1,3,5\},\ E_2 = X\backslash\{1,4,6\},\ E_3 = X\backslash\{2,3,6\},\ E_4 = X\backslash\{2,4,5\},$$
$$E_5 = X\backslash\{7\},\ E_6 = X\backslash\{8\},\ E_7 = X\backslash\{9\}.$$

Then $\tau_{60}(H') = 60\,\tau^*(H')$ for each hypergraph H' arising from H by removing edges. To see this, first observe that if we remove two of the edges E_1,E_2,E_3,E_4 or one of the edges E_5,E_6,E_7, then one of the points of X is in all edges of the remaining hypergraph H', and hence $\nu(H') = 1 = \tau(H')$; in particular $\tau_{60}(H') = 60\tau^*(H')$. So there remains to consider only the hypergraphs H and $H' = (X,E\setminus\{E_1\})$, without loss of generality.

First we consider this last hypergraph. Taking, in (4),

$$t(2) = t(4) = t(6) = 0 \text{ and } t(1) = t(3) = t(5) = t(7) = t(8) = t(9) = 1/5$$

shows $\tau^*(H') \leq 6/5$; taking, in (3),

$$m(E_2) = m(E_3) = m(E_4) = m(E_5) = m(E_6) = m(E_7) = 1/5$$

shows $\nu^*(H') \geq 6/5$. Hence $\nu^*(H') = 6/5 = \tau^*(H')$ and, since these values for t all are multiples of 1/5, $5\tau^*(H') = \tau_5(H')$; this last implies, by (6), $60\tau^*(H') = \tau_{60}(H')$.

Finally look at the hypergraph H itself. Taking

$$t(1) = t(2) = t(3) = t(4) = t(5) = t(6) = 1/12,\ t(7) = t(8) = t(9) = \tfrac{1}{4},$$
$$m(E_1) = m(E_2) = m(E_3) = m(E_4) = 1/8,\ m(E_5) = m(E_6) = m(E_7) = \tfrac{1}{4},$$

shows that $\nu^*(H) = 5/4 = \tau^*(H)$, and that $60\tau^*(H) = \tau_{60}(H)$. These values for m are the only admissible ones attaining the value 5/4; since 1/8 is not a multiple of 1/60 we know that $\nu_{60}(H) \neq 60\nu^*(H)$.

3. PROOFS

We shall prove (13) and (14), from which (11) follows. The proof of (13) is based on the following observation (suggested by the proof methods of LOVÁSZ [3] and EDMONDS & GILES [2]).

<u>LEMMA 1</u>. *Let* P *be a convex polyhedron in* $\mathbb{R}^n$. *If for each vector* $w \in \mathbb{Z}^n$ *the number* $\min\{wx \mid x \in P\}$ *is an integer, or* $\pm\infty$, *then each vertex of* P *has integers as coordinates.*

[wx denotes the usual inner product of w and x.]

<u>PROOF</u>. Suppose P satisfies the premiss of the lemma, and let x_0 be a vertex of P; assume the i-th coordinate of x_0 is not an integer. Since x_0 is a vertex there exists a vector $w \in \mathbb{Z}^n$ such that both $\min\{wx \mid x \in P\}$ and $\min\{w'x \mid x \in P\}$ are attained at x_0, where w' arises from w by adding 1 to

the i-th coordinate of w and leaving the remaining coordinates unchanged. So wx_0 and $w'x_0$ are integers; hence also $w'x_0 - wx_0$, the i-th coordinate of x_0, is an integer, contradicting our assumption. □

EDMONDS & GILES [2] proved that, more generally, the premiss of the lemma implies that each face of P contains integer-valued points. A straightforward adaptation of the proof of lemma 1, or an equally simple replacement of P by $kP = \{kx \mid x \in P\}$, for $k \in \mathbb{Z}$, yields

LEMMA 2. *Let* P *be a convex polyhedron in* $\mathbb{R}^n$. *If for each vector* $w \in \mathbb{Z}^n$ *the number* $\min\{wx \mid x \in P\}$ *is a multiple of* $1/k$, *or* $\pm\infty$, *then all vertices of* P *have* $1/k$-*multiples as coordinates.*

PROOF. As before. □

Evidently, also the Edmonds-Giles extension of lemma 1 can be generalized in a similar way. Now we arrive at the proof of (13).

THEOREM 1. *If* $k\nu^*(H')$ *is an integer for each hypergraph* H' *arising from* H *by multiplication of points then* $k\nu^*(H) = \tau_k(H)$.

PROOF. Suppose H satisfies the conditions. Let P be the convex polyhedron in $\mathbb{R}^X$ consisting of all functions $t\colon X \to \mathbb{R}_+$ such that

$$\sum_{x\in E} t(x) \geq 1$$

for all $E \in \mathcal{E}$. We show that P satisfies the premiss of lemma 2. To this end choose $w \in \mathbb{Z}^X$. It is clear that if one of the coordinates of w is negative then $\min\{wt \mid t \in P\}$ is not finite. So we may assume that $w \in \mathbb{Z}_+^X$; Let H' be the hypergraph arising from H by multiplying every vertex x by w(x). From the definition of multiplication one sees $\nu^*(H') = \tau^*(H') = \min\{wt \mid t \in P\}$, and so this is, by assumption, a multiple of $1/k$. Hence, by lemma 2, each vertex of P has $1/k$-multiples as coordinates; in particular, since each face of P contains a vertex,

$$\tau^*(H) = \min\{\sum_{x\in X} t(x) \mid t \in P\}$$

is attained by some t with 1/k-multiples as values. Therefore

$$k\nu^*(H) = k\tau^*(H) = \tau_k(H). \quad \Box$$

Lovász's result (10) can be extended easily to (14), which is repeated in the following theorem.

THEOREM 2. *If* $\nu_{2k}(H') = 2\nu_k(H')$ *for each hypergraph* H' *arising from* H *by multiplication of points then* $\nu_k(H) = \tau_k(H)$.

PROOF. Adapt straigthforwardly LOVÁSZ's [6] proof of (10). □

4. SOME FURTHER OBSERVATIONS

It can be considered as a main goal of section 3 to give properties of the following sets of nonnegative integers:

(15) $$R = \{k \in \mathbb{Z}_+ \mid \tau_k(H') = k.\tau^*(H') \text{ for each hypergraph } H' \text{ arising from } H \text{ by multiplication of points}\}$$

and

(16) $$S = \{k \in \mathbb{Z}_+ \mid \nu_k(H') = k.\nu^*(H') \text{ for each hypergraph } H' \text{ arising from } H \text{ by multiplication of points}\}.$$

Observe that, by theorem 1,

(17) $$R = \{k \in \mathbb{Z}_+ \mid k\nu^*(H') \text{ is an integer for each hypergraph } H' \text{ arising from } H \text{ by multiplication of points}\}.$$

Therefore $S \subseteq R$ (which is equivalent to (11)). Also define the following set.

(18) $$T = \{k \in \mathbb{Z}_+ \mid \nu_k(H') = \lfloor k\nu^*(H') \rfloor \text{ for each hypergraph } H' \text{ arising from } H \text{ by multiplication of points}\},$$

where $\lfloor x \rfloor$ denotes the lower integer part of a real number x. Clearly $S \subseteq T$; but in general $S \neq T$. E.g., if H has, as edges, all bases of a matroid, then $1 \in T$ (this is the content of Edmonds' matroid base packing theorem), but in general $1 \notin S$. The following theorem gives more properties of and relations between the sets R, S and T, partially derived from results of previous sections.

THEOREM 3.

(i) $\emptyset \neq S = R \cap T$;

(ii) *the set* R *is closed under taking multiples and greatest common divisors;*

(iii) *the set* T, *and hence the set* S *as well, is closed under taking multiples.*

PROOF.

(i) From (16), (17) and (18) above it follows directly that $S = R \cap T$. To show that $S \neq \emptyset$, define the polyhedron

$$P = \{t:X \to \mathbb{R}_+ \mid \sum_{x\in E} t(x) \geq 1 \text{ for all } E \in \mathcal{E}\}. \tag{19}$$

Let $t_1,\ldots,t_m$ be the vertices of P, and, for $i = 1,\ldots,m$, let z_i be the set of all functions $w:X \to \mathbb{R}_+$ such that w as objective function over P attains the minimum in t_i, that is such that $\min\{wt \mid t \in P\}$ is attained in vertex t_i. So each function $w:X \to \mathbb{R}_+$ is in at least one of the Z_i. Note that each Z_i is a closed convex cone. Let, for each $w:X \to \mathbb{Z}_+$, H^w be the hypergraph obtained from H by multiplying each point x by w(x). Then, as in the proof of theorem 1, $\nu^*(H^w) = \min\{wt \mid t \in P\}$. So, for integer-valued $w \in Z_i$, $\nu^*(H^w) = wt_i$, and hence $\nu^*(E^w)$ works additively on the elements of Z_i (for each $i = 1,\ldots,m$).

Now choose $i = 1,\ldots,m$, and let $w_1,\ldots,w_\ell$ be integer-valued vectors in Z_i such that each integer-valued vector in Z_i can be written in the form $\lambda_1 w_1+\ldots+\lambda_\ell w_\ell$ with nonnegative integers $\lambda_1,\ldots,\lambda_\ell$ (this is possible since there are integer-valued vectors $x_1,\ldots,x_r$ such that $Z_i = \{\sum\lambda_j x_j \mid \lambda_j \geq 0\}$; e.g. take as $w_1,\ldots,w_\ell$ all integer-valued vectors contained in $\{\sum\lambda_j x_j \mid 0 \leq \lambda_j \leq 1\}$). Since

$$\nu^*(H^w) = \max\{\sum_{E\in\mathcal{E}} m(E) \mid m:\mathcal{E} \to \mathbb{R}_+; \sum_{E\ni x} m(E) \leq w(x) \text{ for all } x \in X\} \tag{20}$$

and since this function works additively on integer-valued elements of Z_i, each integer-valued vector w in Z_i, being a sum of elements from $w_1,\ldots,w_\ell$, attains the maximum of (20) in the corresponding sum of fuctions $m_1,\ldots,m_\ell$,

attaining the maximum of (20) for $w_1,\dots,w_\ell$. Hence there is an integer k_i such that each integer-valued $w \in Z_i$ attains the maximum of (20) in a function m with $1/k_i$-multiples as values; this means that $k_i\nu^*(H^w) = \nu_{k_i}(H^w)$ for integer-valued $w \in Z_i$. Since there are only a finite number of sets Z_i there is a number k such that $k\nu^*(H^w) = \nu_k(H^w)$ for all $w \in \mathbb{Z}_+^X$, and so $k \in S$, implying the nonemptiness of S. (We thank Lovász for some useful hints.)
(ii) is evident, using (17).
(iii) Using the notation H^w as in the proof of (i) we have that, if $k \in T$ and $\ell \geq 1$, then

$$\nu_{k\ell}(H^w) = \nu_k(H^{\ell w}) = \lfloor k\nu^*(H^{\ell w})\rfloor = \lfloor k\ell\nu^*(H^w)\rfloor$$

for each $w: X \to \mathbb{Z}_+$, and hence $k\ell \in T$. □

We do not know whether S is always closed under taking greatest common divisors. Unlike in previous cases general linear programming techniques will not help to prove this: it is not true that for each rational-valued m×n-matrix A the set

$$(21)\qquad U = \{k \in \mathbb{Z}_+ \mid \text{for each vector } w \in \mathbb{Z}_+^n \text{ the maximum } \max\{\sum_{i=1}^m y_i \mid y \in \mathbb{R}_+^m,\ yA \leq w\} \text{ is attained by a vector } y \text{ with } 1/k\text{-multiples as coordinates}\}$$

is always closed under taking g.c.d.'s. (If we take for A the incidence matrix of H the set U equals S.) If

$$A = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}$$

(A.E. Brouwer's example), then 2 and 3 are elements of U, but 1 is not, showing that U is not closed under taking g.c.d.'s. Clearly, S is closed under taking g.c.d.'s for all hypergraphs H, if and only if U is closed under g.c.d.'s for all (0,1)-matrices A.

The second author conjectured in [8] that if $1 \in R$ then g.c.d.(S) ≤ 2 and gave an example with $1 \in R$ and $2 \notin S$; thus this conjecture would imply that S is not always closed under g.c.d.'s. On the other hand, the first conjecture on p.198 of [9] would imply that $1 \in S$ if g.c.d.(S) = 1.

REFERENCES

[1] BERGE, C., *Balanced hypergraphs and some applications to graph theory*, in: J.N. Srivastava, ed., A Survey of Combinatorial Theory (North-Holland, Amsterdam, 1973), pp.15-23.

[2] EDMONDS, J. & R. GILES, *A min-max relation for submodular functions on graphs*, Annals of Discrete Math. 1 (1977) 185-204.

[3] LOVÁSZ, L., *Normal hypergraphs and the perfect graph conjecture*, Discrete Math. 2 (1972) 253-267.

[4] LOVÁSZ, L., *Minimax theorems for hypergraphs*, Hypergraph Seminar, Lecture Notes in Math. 411 (1974) 111-126.

[5] LOVÁSZ, L., *2-matchings and 2-covers of hypergraphs*, Acta Math. Acad. Sci. Hung. 26 (1975) 433-444.

[6] LOVÁSZ, L., *On two minimax theorems in graph theory*, J. Combinatorial Theory (B) 21 (1976) 96-103.

[7] LOVÁSZ, L., *Certain duality principles in integer programming*, Annals of Discrete Math. 1 (1977) 363-374.

[8] SEYMOUR, P.D., *On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte*, Proc. London Math. Soc., to appear.

[9] SEYMOUR, P.D., *The matroids with the max-flow min-cut property*, J. Combinatorial Theory (B) 23 (1977) 189-222.